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Essentially exact characteristics of turbulent convection in a round tube

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Abstract

The new simplified models and integral formulations recently proposed by Churchill for turbulent flow and convection are utilized herein to compute improved numerical values for the Nusselt number for fully developed convection in a uniformly heated round tube over a wide range of the Reynolds number. The results for two limiting and one intermediate value of the Prandtl number are essentially exact and those for intermediate values appear to be more accurate than previous theoretical predictions. The improved accuracy arises from the use of a theoretically based correlating equation for the turbulent shear stress as well as from the use of the integral formulations. The results are proposed as criteria for evaluation of the accuracy of experimental data, approximate computed values and correlating equations. They also serve to identify those conditions for which improved experimental or theoretically predicted values of the turbulent heat flux density (or turbulent Prandtl number) are critically needed. © 1998 Elsevier Science S.A. All rights reserved.

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1. Introduction

Churchill [1] has recently proposed new and simplified models and integral solutions for fully developed flow and fully developed convection in terms of the local fraction of the transport due to turbulence. A very general and accurate correlating equation for the turbulent shear stress has allowed the use of these integral formulations to derive even more accurate numerical solutions for the velocity distribution and the friction factor for fully developed flow in a round tube and between parallel plates, but the uncertainty in the experimental data and the theoretically predicted values of the local turbulent heat flux density has so far handicapped the development of numerical predictions of corresponding accuracy for the temperature distribution and the Nusselt number. The work reported here is primarily for three particular conditions for which this uncertainty may be avoided, namely the limiting case of $Pr = 0$, the asymptotic case of $Pr \to \infty$, and the special case of $Pr \cong 0.867$ (for which the turbulent Prandtl number is postulated to be equal to the molecular Prandtl number and invariant with the distance from the wall). Approximate results are also presented for intermediate ranges of the Prandtl number for which the dependence on the somewhat uncertain turbulent Prandtl number is quite restrained. These essentially exact solutions are proposed as useful criteria for evaluation of approximate solutions, experimental data and correlations for the overall behavior.

In principal, the new modeling is applicable and valid for fully developed flow in all one-dimensional channels and for all thermal boundary conditions but, in the interests of simplicity, clarity and practicality, the results herein are limited to fully developed convection in fully developed flow in a uniformly heated round tube. Since the development of the models and the derivation of the integral solutions are described in detail by Churchill [1], only those expressions that are essential to understanding are reproduced herein.

In the past, the differential and integral expressions for turbulent flow have generally been formulated in terms of an eddy viscosity or a mixing length for momentum transfer rather than directly in terms of the local turbulent shear stress. These two concepts have often been impugned by purists as arbitrary and empirical but, as shown by Churchill [1], the eddy viscosity and the mixing length themselves are

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both related algebraically to the local turbulent shear stress and thereby are independent of their heuristic diffusional origins. They do have serious shortcomings of a different nature. For example, as discussed by Churchill and Chan [2], the eddy viscosity is unbounded at one point within the fluid and negative over an adjacent finite region in all geometries, including for example circular annuli, in which the total shear stress is unequal on opposing surfaces. However, in a round tube, the only geometry considered in detail herein, the advantage of the new formulations over those in terms of the eddy viscosity is primarily one of simplicity. The mixing length has generally been accorded greater respect than the eddy viscosity by fluid mechanicians, but it is actually inferior in every regard. For example, as apparently first demonstrated by Churchill [1], it is not only unbounded and negative for the same conditions as for the eddy viscosity but is in addition unbounded at the centerline of a round tube and at the central plane of a parallel-plate channel. In all geometries it leads to much more complex formulations than does the eddy viscosity or the turbulent shear stress itself.

The various κ–ε models all function by predicting the eddy viscosity or the mixing length and hence are subject to the same limitations and failures as well as additional ones. The κ–ε–$\overline{u'v'}$ model is free of these shortcomings but requires empirical supplementation for the important region(s) near a wall. All of the models mentioned herein as well as the correlating equation for the turbulent shear stress fail for flow in two-dimensional channels because of the ubiquitous secondary motion.

Differential and integral expressions for turbulent convection have generally been formulated in terms of an eddy conductivity or a mixing length for thermal transport or in terms of an eddy viscosity or mixing length for momentum transport together with the turbulent Prandtl number or its close relative, the total Prandtl number. The failures and shortcomings identified above for the eddy viscosity, the mixing length, the κ–ε model and the κ–ε–$\overline{u'v'}$ model carry over for turbulent convection. Formulations in terms of the turbulent heat flux density or the turbulent shear stress together with the turbulent or total Prandtl number are thereby to be preferred.

How have these shortcomings and outright failures of the eddy viscosity, the mixing length and the κ–ε model, which infest a large fraction of the analytical and numerical results in the literature of turbulent flow and convection, gone undetected or unremarked except in a few instances, such as by, for example, Kjellström and Hedberg [3] and Maubach and Rehme [4]? Apparently because, on the one hand, of an overshadowing scatter in the values of the eddy viscosity and the mixing length as determined by differentiation of experimental values of the time-averaged velocity, and because, on the other hand, the consequent lesser fractional errors in the friction factor and Nusselt number have been small enough to escape notice. In any event, the continued use of the eddy-viscosity, eddy-conductivity, mixing-length and κ–ε models in geometries in which they generate errors of unknown but perhaps significant magnitude (see, for example, Rehme [5]) does not appear to be justifiable. These heuristic models will not be referred to again herein except in comparisons of new numerical solutions with those so-obtained in the past.

The results presented herein were obtained by the numerical evaluation of integrals, an inherently more accurate procedure than numerical integration of partial or even ordinary differential equations or of determination of eigenvalues and eigencoefficients for a series solution.

2. Momentum transfer

The time-averaged force-momentum balance for fully developed turbulent motion of a fluid with invariant physical properties in a smooth round tube may be expressed as

$$\tau = \tau_w\left(1 - \frac{y}{a}\right) = \mu\frac{du}{dy} - \rho(\overline{u'v'}) \tag{1}$$

which may be rewritten in dimensionless form as

$$\left(1 - \frac{y^+}{a^+}\right)\left[1 - (\overline{u'v'})^{++}\right] = \frac{du^+}{dy^+} \tag{2}$$

where

$$u^+ = u\left(\frac{\rho}{\tau_w}\right)^{1/2}$$

$$y^+ = \frac{y(\rho\tau_w)^{1/2}}{\mu}$$

$$a^+ = \frac{a(\rho\tau_w)^{1/2}}{\mu}$$

and

$$(\overline{u'v'})^{++} = -\frac{\rho(\overline{u'v'})}{\tau}$$

The dimensionless quantity $(\overline{u'v'})^{++}$ may obviously be interpreted as the fraction of the local shear stress due to the turbulent fluctuations.

Introducing $R = 1-(y^+/a^+)$ and integrating Eq. (2) formally results in

$$u^+ = \frac{a^+}{2}\int_{R^2}^{1}\left[1 - (\overline{u'v'})^{++}\right]dR^2 \tag{3}$$

It follows, by means of integration by parts, that

$$\left(\frac{2}{f}\right)^{1/2} = u_m^+ = \int_0^1 u^+ dR^2 = \frac{a^+}{4}\int_0^1\left[1 - (\overline{u'v'})^{++}\right]dR^4 \tag{4}$$

In both Eqs. (3) and (4) the term representing the contribution of the turbulent fluctuations is simply subtracted from that corresponding to purely viscous flow. Neither this deductibility nor the possibility of integration by parts are so apparent in formulations based on the eddy viscosity or the mixing length. Indeed, even the possibility of expressing u_m^+ in terms of a single integral of a function of the eddy viscosity, or the mixing length has apparently never been recognized.

Churchill and Chan [6] devised the following correlating equation for the local turbulent shear stress in fully developed flow in a smooth round tube:

$$(\overline{u'v'})^+ = \left(\left[0.7\left(\frac{y^+}{10}\right)^3\right]^{-8/7} + \left[\left(1-\frac{y^+}{a^+}\right)\left|\exp\left\{-\frac{2.5}{y^+}\right\} - \frac{2.5}{a^+}\left(1+\frac{4y^+}{a^+}\right)\right|\right]^{-8/7}\right)^{-7/8} \quad (5)$$

The functionality of the individual terms in Eq. (5) is based on an asymptotic expression for $\overline{u'v'}$ for the region very near the wall and asymptotic expressions for the time-averaged velocity near the centerline and in the intermediate, semi-logarithmic regime of 'overlap' ($30 < y^+ < 0.1a^+$). The expressions for $(\overline{u'v'})^+$ for the latter two regions follow from Eq. (2) as reexpressed in terms of $(\overline{u'v'})^+ = (1-(y^+/a^+))(\overline{u'v'})^{++}$. The coefficient of 0.7 is based on experimental measurements of $\overline{u'v'}$ as well as on direct numerical simulations by several investigators. The coefficient of 2.5 is based on the correlating equation of Nikuradse [7] for his own experimental measurements of the time-mean velocity in the turbulent core, and the coefficient of 4.0 on the magnitude of the turbulent 'wake' at the centerline as inferred by Reichardt [8] from the measurements of the time-mean velocity by Nikuradse and others. the exponential-mean of the terms for small and large values of y^+ is based on the generalized correlating equation proposed by Churchill and Usagi [9], while the value of $-8/7$ for the arbitrary combining exponent was chosen on the basis of experimental data for both $(\overline{u'v'})^+$ and u^+. Eq. (5) was shown to represent the experimental data of Wei and Willmarth [10] for $\overline{u'v'}$ for large values of y^+ and that of Eckelmann [11] as well as the computed values by direct numerical simulation of several investigators for small values. Values of u^+ and u_m^+ computed from Eqs. (3) and (4) respectively, using values of $(\overline{u'v'})^+$ from Eq. (5) were shown to agree very well with the experimental data of Nikuradse and others for $a^+ > 300$. The small but observable discrepancies in f for smaller values of a^+ are presumed to be a consequence of the physical disappearance of the semilogarithmic regime of the velocity even though it remains a component of Eq. (5).

Zagarola [12] has recently obtained extensive experimental data for the friction factor and for the velocity distribution in the turbulent core at very high Reynolds numbers in a very long smooth tube. In view of their precision and the use of moden instrumentation, these values are presumed to be more accurate than those of Nikuradse. They suggest that the coefficients of 2.5 and 4.0 in Eq. (5) should be replaced by 2.295 and 6.95, respectively. In terms of $(\overline{u'v'})^{++}$ rather than $(\overline{u'v'})^+$, Eq. (5) then becomes

$$(\overline{u'v'})^{++} = \left(\left[0.7\left(\frac{y^+}{10}\right)^3\right]^{-8/7} + \left|\exp\left\{\frac{-2.294}{y^+}\right\} - \frac{2.294}{a^+}\left(1+\frac{6.95y^+}{a^+}\right)\right|^{-8/7}\right)^{-7/8} \quad (6)$$

The factor of $1-(y^+/a^+)$ that arises in the first term on the righthand side of this expression by virtue of the transformation of the dependent variable has been dropped in the interests of simplicity and possibly accuracy. The integrations of $(\overline{u'v'})^{++}$ that are carried out to evaluate u^+ and u_m^+ by means of Eqs. (3) and (4), respectively, have the effect of reducing the small errors associated with Eq. (6). The numerical values of the friction factor that are so-computed may be represented almost exactly by the following empirical expression:

$$\left(\frac{2}{f}\right)^{1/2} = 3.296 - \frac{161.2}{a^+} + \left(\frac{47.6}{a^+}\right)^2 + 2.294\ln\{a^+\} \quad (7)$$

Furthermore, Eq. (7) predicts the values of the friction factor determined experimentally by Zagarola even more accurately than his own correlating equation. Eq. (7) may be expressed in terms of Re_D simply by substituting $Re_D\ (f/8)^{1/2}$ for a^+, but the original form is to be preferred in terms of explicitness and simplicity. Although the terms in $(a^+)^{-1}$ and $(a^+)^{-2}$ have generally been overlooked in the past, they are a necessary consequence of $u^+ \to y^+$ as $y^+ \to 0$. They are significant numerically for values of a^+ approaching the lower limit of fully turbulent flow. The numerical effect on the friction factor of the terms in Eq. (6) that represent the wake, namely those involving a^+, is significant and invariant for all values of a^+. Eq. (6) and thereby u^+, as calculated from Eq. (3), and f, as calculated from Eq. (4) or Eq. (7), are somewhat uncertain for $Re_D < 10^4$ ($a^+ < 300$) owing to the aforementioned physical disappearance of the regime of 'overlap'.

2.1. Energy transfer

The energy balance corresponding to Eq. (1), but with the additional assumption of negligible viscous dissipation, is

$$j = -k\frac{dT}{dy} + \rho c(\overline{T'v'}) \quad (8)$$

which, by analogy to Eq. (2), may be rewritten in dimensionless form as

$$\frac{j}{j_w}\left[1 - (\overline{T'v'})^{++}\right] = \frac{dT^+}{dy^+} \quad (9)$$

where

$$T^+ = \frac{k(T_w - T)(\tau_w\rho)^{1/2}}{\mu j_w}$$

and

$$(\overline{T'v'})^{++} = \frac{\rho c(\overline{T'v'})}{j}$$

It is convenient to reexpress Eq. (9) as

$$\frac{j}{j_{\mathrm{w}}}\left[1-(\overline{u'v'})^{++}\right]\frac{\mathrm{Pr_T}}{\mathrm{Pr}} = \frac{\mathrm{d}T^+}{\mathrm{d}y^+} \tag{10}$$

where

$$\frac{\mathrm{Pr_T}}{\mathrm{Pr}} = \frac{c(\mu+\mu_{\mathrm{t}})}{k+k_{\mathrm{t}}}\left(\frac{k}{c\mu}\right) = \frac{1-(\overline{T'v'})^{++}}{1-(\overline{u'v'})^{++}} \tag{11}$$

The presence of μ_{t} and k_{t} in the central term of Eq. (11) implies that the eddy-diffusivity model of Boussinesq [13] was involved in deriving Eq. (10). However, from the right-most term of Eq. (11) it may also be inferred that $\mathrm{Pr_T}$ depends only on $(\overline{T'v'})^{++}$ and $(\overline{u'v'})^{++}$ and hence is independent of the eddy diffusional concept itself. An alternative to Eq. (10) but subject to the same inferences is

$$\frac{j}{j_{\mathrm{w}}} = \left[1+\frac{\mathrm{Pr_t}}{\mathrm{Pr}}\left(\frac{(\overline{u'v'})^{++}}{1-(\overline{u'v'})^{++}}\right)\right]\frac{\mathrm{d}T^+}{\mathrm{d}y^+} \tag{12}$$

where

$$\frac{\mathrm{Pr_t}}{\mathrm{Pr}} = \left(\frac{c\mu_{\mathrm{t}}}{k_{\mathrm{t}}}\right)\left(\frac{k}{c\mu}\right) = \frac{(\overline{u'v'})^{++}}{(\overline{T'v'})^{++}}\left(\frac{1-(\overline{T'v'})^{++}}{1-(\overline{u'v'})^{++}}\right) \tag{13}$$

Herein $\mathrm{Pr_T}/\mathrm{Pr}$ and $\mathrm{Pr_t}/\mathrm{Pr}$ should be interpreted only as symbols representing the right-most terms of Eqs. (11) and (13), respectively.

The following exact relationship between $\mathrm{Pr_T}$ and $\mathrm{Pr_t}$ may be derived from Eqs. (10) and (12) or Eqs. (11) and (13):

$$\frac{1}{\mathrm{Pr_T}} = \frac{(\overline{u'v'})^{++}}{\mathrm{Pr_t}} + \frac{1-(\overline{u'v'})^{++}}{\mathrm{Pr}} \tag{14}$$

$\mathrm{Pr_t}$ and $\mathrm{Pr_T}$, and hence $(\overline{T'v'})^{++}$ are presumed (see Churchill [1]) to be dependent only on Pr and $(\overline{u'v'})^{++}$ or the equivalent and to be independent of geometry and the thermal boundary condition(s), but the evidence in support of that premise is less than conclusive and the dependence itself is quite uncertain, particularly for Pr < 0.7. Definitive general solutions for $T^+\{y^+, a^+, \mathrm{Pr}\}$ and $\mathrm{Nu}_D\,\{a^+, \mathrm{Pr}\}$, starting from Eqs. (9), (10) and (12) must await resolution of these uncertainties. Hence, attention herein is first focused on those few particular cases for which the dependence of T^+ and Nu_D on $\mathrm{Pr_t}$ and $\mathrm{Pr_T}$ is absent, and then secondarily on those ranges in which the effect of these uncertainties is minimal.

Eq. (9) appears to be simpler than Eq. (10), which is in turn simpler than Eq. (12). However, the best choice as a starting point for determination of the Nusselt number depends on the relative invariance of $(\overline{T'v'})^{++}$, $\mathrm{Pr_T}$ and $\mathrm{Pr_t}$ with y^+ and thereby depends on the value or range of Pr. For that reason, both Eqs. (10) and (12) but not Eq. (9) will be utilized herein.

As may be inferred from Eq. (2), the variation of the total shear stress across a round tube is given exactly by $\tau/\tau_{\mathrm{w}} = 1-(y^+/a^+) = R$ and is thus independent of a^+ (or Re_D). On the other hand, the variation of the relative total heat flux density, j/j_{w}, is not only unknown a priori, but depends complexly on a^+ (or Re_D) for uniform heating, and on Pr as well for other thermal boundary conditions. The postulate that j/j_{w} is invariant or equal to τ/τ_{w} is the principal source of error in many prior solutions for Nu_D [see Churchill [14]]. The improved accuracy of the numerical results presented in this paper is to a considerable degree due to the use of essentially exact values of j/j_{w}.

2.2. Local heat flux density

For a uniform heat flux density from the wall, fully developed flow, complete thermal development, negligible viscous dissipation, and negligible axial transport of energy by thermal conduction and the turbulent fluctuations, the local heat flux density may be shown to be given exactly by

$$\frac{j}{j_{\mathrm{w}}} = \frac{1}{R}\int_0^{R^2}\left(\frac{u^+}{u_{\mathrm{m}}^+}\right)\mathrm{d}R^2 \tag{15}$$

It is evident from Eq. (15) that j/j_{w} is equal to τ/τ_{w} only for the hypothetical condition of plug flow, and depends on y^+ and a^+ (or Re_D) by virtue of $u^+\{y^+, a^+\}$ and $u_{\mathrm{m}}^+\{a^+\}$. It is worthy of note and perhaps surprising that j/j_{w} is independent of Pr for uniform heating. This independence does not occur for any other thermal boundary condition. Since j/j_{w} might be expected to deviate only moderately from τ/τ_{w}, it is convenient to express this dependence in terms of a perturbation γ defined by

$$\frac{j}{j_{\mathrm{w}}} = (1+\gamma)\left(\frac{\tau}{\tau_{\mathrm{w}}}\right) = (1+\gamma)R \tag{16}$$

Substituting in Eq. (15) for j/j_{w} from Eq. (16), for u^+ from Eq. (4) and for u_{m}^+ from Eq. (4) gives after integration by parts and rearrangement

$$\gamma = \frac{((1-R^2)/R^2)\int_0^{R^4}[1-(\overline{u'v'})^{++}]\mathrm{d}R^4 + \int_{R^4}^1((1-R^2)/R^2)[1-(\overline{u'v'})^{++}]\mathrm{d}R^4}{\int_0^1[1-(\overline{u'v'})^{++}]\mathrm{d}R^4} \tag{17}$$

Illustrative values of γ determined by numerical evaluation of the integrals in Eq. (17) using Eq. (6) for $(\overline{u'v'})^{++}$ are listed in Table 1. γ is seen to increase monotonically from zero at the wall to a maximum value at the centerline that decreases with increasing a^+ (or Re_D). Since u^+ approaches u_{c}^+ as $R\to 0$, this maximum value may be inferred from Eq. (15) to be equal to

Table 1
The parameter $\gamma = j\tau_w/j_w\tau - 1$ for a uniformly heated round tube

y^+	a^+				
	500	1000	2000	5000	10,000
0	0	0	0	0	0
1	0.00391	0.001959	0.00009807	0.0003929	0.0001967
2	0.00761	0.003815	0.001913	0.000768	0.0003585
3	0.06108	0.005567	0.002797	0.001125	0.0005650
4	0.01434	0.007216	0.003632	0.001465	0.0007368
5	0.01737	0.008764	0.004421	0.001788	0.0009004
10	0.02969	0.01516	0.007738	0.002094	0.001612
20	0.04615	0.02380	0.01256	0.002387	0.002705
30	0.05875	0.03070	0.01647	0.003171	0.003646
y^+/a^+					
0.01	0.01737	0.01516	0.01256		
0.10	0.07960	0.0661	0.05856	0.05137	0.04749
0.20	0.1210	0.103	0.09251	0.08201	0.07607
0.50	0.1845	0.183	0.1660	0.10725	0.09962
0.90	0.2720	0.239	0.2171	0.1944	0.1808
1.00	0.2748	0.2424	0.2195	0.1966	0.1828

$$\gamma_{R=0} = \frac{u_c^+}{u_m^+} - 1 = \frac{2\int_0^1 [1 - (\overline{u'v'})^{++}]\mathrm{d}R^2}{\int_0^1 [1 - (\overline{u'v'})^{++}]\mathrm{d}R^4} - 1 \tag{18}$$

In view of the smoothing due to the integrations by means of which γ was evaluated, the listed values are presumed to be more accurate than the predicted values of $(\overline{u'v'})^{++}$ and hence almost exact. Since γ appears in subsequent relationships only in the form of $1+\gamma$, the fractional uncertainty is even further reduced in that quantity.

2.3. Temperature distribution and Nusselt number

Substituting for j/j_w from Eq. (16) and then integrating Eq. (10)formally gives

$$T^+ = \frac{a^+}{2}\int_{R^2}^1 (1+\gamma)[1 - (\overline{u'v'})^{++}]\left(\frac{\mathrm{Pr_T}}{\mathrm{Pr}}\right)\mathrm{d}R^2 \tag{19}$$

By virtue of Eqs. (15) and (19), and integration by parts,

$$T_m^+ \equiv \int_0^1 \left(\frac{u^+}{u_m^+}\right)T^+\mathrm{d}R^2 = \frac{a^+}{4}\int_0^1 (1+\gamma)^2[1 - (\overline{u'v'})^{++}]\left(\frac{\mathrm{Pr_T}}{\mathrm{Pr}}\right)\mathrm{d}R^4 \tag{20}$$

From which it follows that

$$\mathrm{Nu}_D = \frac{2a^+}{T_m^+} = \frac{8}{\int_0^1 (1+\gamma)^2[1 - (\overline{u'v'})^{++}](\mathrm{Pr_T}/\mathrm{Pr})\mathrm{d}R^4} \tag{21}$$

Starting from Eq. (12) rather than from Eq. (10), or simply substituting for $\mathrm{Pr_T}/\mathrm{Pr}$ in Eq. (21) from Eq. (14), results in the corresponding expression in terms of $\mathrm{Pr_t}$ rather than $\mathrm{Pr_T}$, namely,

$$\mathrm{Nu}_D = \frac{8}{\int_0^1 ((1+\gamma)^2\mathrm{d}R^4)/(1+(\mathrm{Pr}/\mathrm{Pr_t})((\overline{u'v'})^{++}/(1-(\overline{u'v'})^{++})))} \tag{22}$$

2.4. Essentially exact values of Nu_D

The primary source of error in the evaluation of Nu_D from Eq. (21) or Eq. (22) is that associated with the dependence of $\mathrm{Pr_T}$ and $\mathrm{Pr_t}$ on $(\overline{u'v'})^{++}$ and Pr. Accordingly, Nu_D will first be evaluated for the two values of Pr, namely $\mathrm{Pr}=0$ and $\mathrm{Pr}=\mathrm{Pr_t}=\mathrm{Pr_T}$, for which that dependence is not required, and for one, namely $\mathrm{Pr}\to\infty$, for which it is known with reasonable certainty.

$\mathrm{Pr}=0$

For this limiting condition, $(\overline{T'v'})^{++}\to 0$, and Eq. (22) may be seen to reduce to

$$\mathrm{Nu}_D = \frac{8}{\int_0^1 (1+\gamma)^2\mathrm{d}R^4} = \frac{8}{1+\varepsilon_1} \tag{23}$$

where

$$1+\varepsilon_1 = \int_0^1 (1+\gamma)^2\mathrm{d}R^4 \tag{24}$$

The value of ε_1 as determined from Eq. (24) is presumed to be more accurate than the values of γ from which it is calculated owing to this latter integration. The effect of any uncertain in γ on Nu_D is further reduced by the appearance of ε_1 as a perturbation with respect to unity. The quantity ε_1 is a direct consequence of the deviation of the heat flux density ratio from the shear stress ratio but may be interpreted alternatively as a measure of the effect of the velocity distribution, since for plug flow $\mathrm{Nu}_D = 8$. Thus in the limit of $\mathrm{Pr}=0$, the fluctuations in $u'v'$ affect the heat transfer even though those in $T'v'$ do not. For $\mathrm{Pr}\to 0$, Eq. (14) may be inferred to reduce to

$$\frac{\mathrm{Pr}}{\mathrm{Pr_T}} = 1 - (\overline{u'v'})^{++} \tag{25}$$

by virtue of which Eq. (21) also reduces to Eq. (23).

It follows from Eq. (19) that

$$T^+ = \frac{a^+}{2}\int_{R^2}^1 (1+\gamma)\mathrm{d}R^2 \tag{26}$$

and

$$T_c^+ = \frac{a^+}{2}\int_0^1 (1+\gamma)\mathrm{d}R^2 \tag{27}$$

and from Eq. (20) that

$$T_m^+ = \frac{a^+}{4}\int_0^1 (1+\gamma)^2\mathrm{d}R^4 \tag{28}$$

Table 2
The thermal characteristics of fully developed convection in a uniformly heated round tube for $\mathrm{Pr} = 0$

a^+	$\mathrm{Re}_D \times 10^{-3}$	Nu_D			ε_1	T_m^+/T_m^+	u_c^+/u_m^+
		Kays and Leung[a]	Notter and Sleicher[a]	New or exact values			
<63	<2	–	–	4.364	0.8333	1.4583	2.0000
500	16.90	6.490	6.82	6.447	0.2409	1.858	1.2704
1000	36.58	6.695	6.935	6.647	0.2036	1.881	1.2432
2000	80.00	6.845	7.03	6.784	0.1793	1.896	1.2199
5000	221.8	6.895	7.175	6.911	0.1576	1.909	1.1970
10,000	475.9	6.995	7.30	6.985	0.1454	1.916	1.1829
Plug flow	–	–	–	8.000	0.0000	1.250	1.000

[a]Interpolated with respect to Re_D.

Numerical values of Nu_D for $\mathrm{Pr} = 0$, as evaluated from Eq. (23) using Eq. (17) for γ and Eq. (6) for $(\overline{u'v'})^{++}$, are listed in Table 2 for a series of values of a^+. The corresponding values of $\mathrm{Re}_D = 2a^+u_m^+, \varepsilon_1$ and T_c^+/T_m^+ are included in the listing. These values of Nu_D provide a lower bound for the turbulent regime with uniform heating. Those for the turbulent regime properly fall between 8 and $48/11 = 4.364$, the values for plug and laminar flow, respectively, as do the values of T_c^+/T_m^+ between 2 and $18/11 = 1.637$.

The prior theoretical values included in Table 2 are discussed subsequently.

$\mathrm{Pr}_t = \mathrm{Pr}_T = \mathrm{Pr}$

The turbulent Prandtl number Pr_t has been found experimentally and from a theoretical expression to be equal to the molecular Prandtl number Pr for all values of y^+ when the latter is equal to about 0.867. It follows from Eq. (14) that the total Prandtl number Pr_T then has the same value as well and from Eqs. (11) and (13) that $(\overline{T'v'})^{++} = (\overline{u'v'})^{++}$. For this particular condition, both Eqs. (21) and (22) reduce to

$$\mathrm{Nu}_D = \frac{8}{\int_0^1 (1+\gamma)^2[1-(\overline{u'v'})^{++}]\mathrm{d}R^4} \tag{29}$$

which, just as Eq. (23), has the merit of independence from Pr_t or Pr_T. By virtue of Eq. (4), this expression may be rewritten as

$$\mathrm{Nu}_D = \frac{\mathrm{Re}f/2}{1+\varepsilon_2} \tag{30}$$

where $1+\varepsilon_2$ is equal to the integrated-mean value over R^4 of $(1+\gamma)^2$ weighted by $1-(\overline{u'v'})^{++}$, that is,

$$1+\varepsilon_2 = \frac{\int_0^1 (1+\gamma)^2[1-(\overline{u'v'})^{++}]\mathrm{d}R^4}{\int_0^1 [1-(\overline{u'v'})^{++}]\mathrm{d}R^4} \tag{31}$$

The failure of the Reynolds analogy, as measured quantitatively by ε_2, is thus a consequence of the implicit postulate of $j/j_w = \tau/\tau_w$, i.e., of $\gamma = 0$. The common but erroneous presumption that the Reynolds analogy is valid at $\mathrm{Pr} = 1$, if at all, implies that $\mathrm{Pr}_t = \mathrm{Pr}_T = 1$ at $\mathrm{Pr} = 1$. Insofar as the Nusselt number is proportional to Pr in this range of Pr, the Reynolds analogy in its original form:

$$\mathrm{Nu}_D = \mathrm{Pr}\,\mathrm{Re}_D(f/2) \tag{32}$$

may be implied to apply approximately at $\mathrm{Pr} = 1+\varepsilon_2$, a quantity that varies slightly with a^+ (or Re_D) by virtue of the finite values of γ.

It follows from Eqs. (19) and (20) that for $\mathrm{Pr} = \mathrm{Pr}_t = \mathrm{Pr}_T$,

$$\frac{T_c^+}{T_m^+} = \frac{2\int_0^1 (1+\gamma)[1-(\overline{u'v'})^{++}]\mathrm{d}R^2}{\int_0^1 (1+\gamma)^2[1-(\overline{u'v'})^{++}]\mathrm{d}R^4} \tag{33}$$

Numerical values of Nu_D, ε_2 and T_c^+/T_m^+ computed from Eqs. (29), (31) and (33), using $(\overline{u'v'})^{++}$ from Eq. (6) and γ from Eq. (17), are listed in Table 3 for a series of values of a^+ and the corresponding values of Re_D. These values have no prior counterpart in the literature. They are perhaps the most important results of this investigation in that they provide an essentially exact basis for the evaluation of experimental data, correlating equations and approximate

Table 3
The thermal characteristics for fully developed convection in a uniformly heated round tube for $\mathrm{Pr} = \mathrm{Pr}_t = 0.867$

a^+	$\mathrm{Re}_D \times 10^{-3}$	Nu_D			ε_2	T_c^+/T_m^+
		Kays and Leng[a]	Notter and Sleicher[a]	New or exact value		
<63	<2	–	–	4.364	0.8333	1.4583
500	16.90	52.7	55.4	55.12	0.0736	1.2463
1000	36.58	97.1	101.9	102.0	0.0725	1.2269
2000	80.00	177.7	188.2	189.5	0.0550	1.2106
5000	221.8	401.7	428.0	433.0	0.0413	1.1923
10,000	475.9	749.8	771.6	812.4	0.0346	1.1803

[a]Interpolated with respect to both Re_D and Pr.

computed values for this particular intermediate value of Pr and all values of a^+ (or Re_D).

$\underline{P}r \to \infty$

For very large values of the Prandtl number the temperature profile develops almost wholly very near the wall where $j/j_w \to 1$ and

$$(\overline{u'v'})^{++} \to \propto (y^+)^3 \tag{34}$$

This expression with $\propto \cong 7 \times 10^{-4}$, as implied by Eq. (6), has both experimental and theoretical credentials. The experimental data for heat transfer at large values of Pr indicate that near the wall $\mathrm{Pr_t}$ is essentially invariant with respect to both y^+ and a^+, and has a value of about 0.85. On the other hand, insofar as $\mathrm{Pr_t}$ is invariant with y^+, $\mathrm{Pr_T}$ must vary greatly according to Eq. (14). Eq. (12) is therefore a more convenient starting point than Eq. (10) to derive an expression for Nu_D for large values of Pr. Eq. (10) may, after substituting $(\overline{u'v'})^{++}$ from Eq. (6) and making the postulate that $\mathrm{Pr_t}$ and j are invariant with y^+, be integrated analytically to obtain an expression for T_c^+ that is equivalent to

$$\mathrm{Nu}_D = \frac{3^{3/2}}{2\pi}\left(1-\frac{\mathrm{Pr_t}}{\mathrm{Pr}}\right)^{4/3}\left(\frac{\propto \mathrm{Pr}}{\mathrm{Pr_t}}\right)^{1/3}\left(\frac{T_c^+}{T_m^+}\right)\mathrm{Re}_D\left(\frac{f}{2}\right)^{1/2} \tag{35}$$

For $\propto = 7 \times 10^{-4}$ and $\mathrm{Pr_t} = 0.85$, and from the recognition that $(1-(\mathrm{Pr_t}/\mathrm{Pr}))^{4/3}$ and T_c^+/T_m^+ both approach unity as $\mathrm{Pr} \to \infty$, Eq. (35) may be reduced to

$$\mathrm{Nu}_D = 0.078\mathrm{Pr}^{1/3}\mathrm{Re}_D\left(\frac{f}{2}\right)^{1/2} \tag{36}$$

The equivalent of Eq. (35) was apparently first derived by Churchill [15] but the equivalent of Eq. (36) has been derived by Petukhov [16] and others. Eq. (36) has ample confirmation as indicated by the plot of experimental data in Figure 1 of Churchill [17] and the numerical solution of Notter and Sleicher [16].

2.5. Approximate values of Nu_D

Jischa and Rieke [19] and others have correlated experimental values of the turbulent Prandtl number in the turbulent core $(30 < y^+ \le a^+)$ for $\mathrm{Pr} \ge 0.7$ with empirical expressions such as

$$\mathrm{Pr_t} = 0.85 + \frac{0.015}{\mathrm{Pr}} \tag{37}$$

Despite these nominal limitations, Eq. (37) may be speculated to provide an adequate approximation for all y^+ and Pr insofar as the determination of Nu_D is concerned. This extended applicability is suggested by the form of the denominator of the integral in Eq. (22). $(\mathrm{Pr}/\mathrm{Pr_t})((\overline{u'v'})^{++}/(1-(\overline{u'v'})^{++}))$ is small with respect to unity near the wall for moderate values of Pr by virtue of small values of $(\overline{u'v'})^{++}$ and is also small for all y^+ for small values of Pr by virtue of large values of $\mathrm{Pr_t}$ as well as small values of Pr.

Values of Nu_D computed from Eq. (22) using $\mathrm{Pr_t}$ from Eq. (37) are listed in Table 4. Values of $\mathrm{PrRe}_D(f/2)$ are included for comparison for small values of Pr, while the values of Nu_D for large values of Pr are divided by $0.07343(\mathrm{Pr}/\mathrm{Pr_t})^{1/3}\ \mathrm{Re}_D\ (f/2)^{1/2}$ to reduce their magnitude and indicate their deviation from the asymptotic expression.

It may be noted that Eq. (37) was utilized implicitly in the derivations for $\mathrm{Pr} = \mathrm{Pr_t}$ and $\mathrm{Pr} \to \infty$.

3. Evaluation of accuracy

The differential, integral and algebraic expressions presented herein, with the exception of Eqs. (6) and (7), the

Table 4
Approximate thermal characteristics for fully developed convection in a uniformly heated round tube

Small values of Pr a^+	$\mathrm{Re}_D \times 10^{-3}$	Nu_D						
		0	0.001	0.01	0.1	0.7	0.867	$\mathrm{Re}_D\,f/2$
500	16.90	6.447	6.473	7.060	16.79	49.13	55.12	59.18
1000	36.58	6.647	6.682	7.917	26.85	90.02	102.0	109.4
2000	80.00	6.784	6.828	9.305	44.56	166.0	189.5	200.0
5000	221.8	6.911	7.016	12.923	90.87	375.4	433.0	450.9
10,000	475.9	6.985	7.190	18.217	159.4	700.7	812.4	840.5

Large values of Pr		$\mathrm{Nu}_D/0.07343(\mathrm{Pr}/\mathrm{Pr_t})^{1/3}\mathrm{Re}_D^{(1/2)}$					
		1.0	10.0	100.0	1000	10,000	∞
500	16.9	0.7572	0.9939	0.9846	0.9969	0.9998	1.000
1000	36.58	0.7389	0.9287	0.9803	0.9968	0.9996	1.000
2000	80.00	0.6569	0.8978	0.9758	0.9951	0.9986	1.000
5000	221.8	0.6038	0.8758	0.9704	0.9940	0.9985	1.000
10,000	475.9	0.5686	0.8592	0.9658	0.9930	0.9987	1.000

possible exception of Eqs. (34)–(36), and of course Eq. (37), are all exact insofar as the conditions of fully developed turbulent flow and fully developed convection are achieved asymptotically, and viscous dissipation and the variation of the physical properties with pressure and temperature may be neglected.

For u^+ and u_m^+, and therefore for u_c^+ and f, the only sources of error are the expression used for $(\overline{u'v'})^{++}$, namely Eq. (6), and the numerical process itself. Since the latter only involves the straightforward evaluation of well-behaved integrals, the computational error may readily be reduced below any chosen level. For $\mathrm{Pr} = 0$, the same considerations apply to γ, T^+ and T_m^+ and thereby to T_c^+ and Nu_D. The results for $\mathrm{Pr} = \mathrm{Pr_t} = \mathrm{Pr_T}$ additionally imply that the latter quantities are invariant with respect to y^+. On the other hand, Nu_D for $\mathrm{Pr} \to \infty$, as given by Eq. (35), depends on the functional validity of Eq. (34) and the postulate that $\mathrm{Pr_t}$ approaches a limiting value as $y^+ \to 0$. Eq. (36) incorporates additionally the numerical coefficients of $\alpha = 7 \times 10^{-4}$ and $\mathrm{Pr_t} = 0.85$. These sources of possible error as well as the validity of the results obtained by means of Eq. (37) will now be examined in some detail.

3.1. Uncertainty arising from Eq. (6)

Eq. (6), which was used for $(\overline{u'v'})^{++}$ for all of the numerical calculations herein, is empirical and subject to uncertainty in its form as well as in its several coefficients and exponents. Consider first the asymptotic behavior of $(\overline{u'v'})^{++}$ for $y^+ \to 0$ that appears as the first term on the right-hand side of Eq. (6) and is given explicitly by Eq. (34). The third-power dependence, which follows from various asymptotic analyses, has strong support experimentally and from direct numerical simulations as does the coefficient of 7×10^{-4}. Any related uncertainty in u^+, u_m^+, T^+, T_m^+ may be presumed to be negligible.

On the other hand, the postulated dependence of $(\overline{u'v'})^{++}$ on y^+ and a^+ for large values of y^+, as given by the term within the absolute value signs of Eq. (6), has a semitheoretical structure but completely empirical coefficients. The numerical coefficient of 2.294 that appears twice in this expression corresponds to the slope in plots by Zagarola of his values of u^+ versus $\ln\{y^+\}$ in the range of $30 < y^+ < 0.1a^+$ as well as in plots of $u_m^+ = (2/f)^{1/2}$ versus $\ln\{a^+\}$ for $a^+ > 1000$. The term $(2.295/a^+)(1 + (6.95y^+/a^+))$ is a consequence of the postulate of $\mathrm{d}u^+/\mathrm{d}y^+ = 0$ at $y^+ = a^+$, the postulate that $u_c^+ - u^+$ is proportional to $(1-(y^+/a^+))^2$ near the centerline and the observation by Zagarola [12] of a limiting deviation from the semilogarithmic regime due to the wake of 1.51 in u^+.

The absolute value signs of Eq. (6) and the approximation of $1-(2.294/y^+)$ by $\exp\{-2.294/y^+\}$ therein are both merely mathematical contrivances to avoid negative and unbounded values for that term in the range of small values of y^+ for which it contributes negligibly to the prediction of $(\overline{u'v'})^{++}$.

The exponential-mean form of Eq. (6) and the exponent of $-8/7$ are arbitrary but the predicted values of $(\overline{u'v'})^{++}$ are relatively insensitive to the value of that exponent.

On the basis of the above analysis it seems reasonable to conclude that the values of $(\overline{u'v'})^{++}$ predicted by Eq. (6) are adequate for computation of $\gamma, T^+, T_c^+, T_m^+$ and Nu_D for $a^+ > 300$.

3.2. Uncertainty in γ, T^+ and Nu_D for $Pr=0$ and $Pr=Pr_t=Pr_T$

For uniform heating, the small error in the predictions of $(\overline{u'v'})^{++}$ by Eq. (6) is greatly reduced in γ by the evaluation of the integrals within Eq. (17). This error is further reduced in T^+ by the integration posed by Eq. (19) and even further in T_m^+ and Nu_D by the integration posed by Eq. (20). The convergence of these several integrations was tested by the successive use of a greater number of increments. On this basis the listed values of γ are presumed to be reasonably accurate, those of T^+ to be even more accurate and those of Nu_D to be essentially exact.

3.3. Uncertainty in Nu_D for $Pr \to \infty$

The uncertainty in Nu_D for $\mathrm{Pr} \to \infty$, as given by Eq. (36), has an entirely different origin than that for $\mathrm{Pr} = 0$ and $\mathrm{Pr} = \mathrm{Pr_t} = \mathrm{Pr_T}$. It arises only from that of Eq. (34) and the postulate of a limiting value of $\mathrm{Pr_t}$ as $y^+ \to 0$. The support for the third-power dependence of $(\overline{u'v'})^{++}$ on y^+ and for the value of 7×10^{-4} for the coefficient $\propto$ have already been described. The existence of a limiting value for $\mathrm{Pr_t}$ for $y^+ \to 0$ and the limiting value of 0.85 itself, as well as the resulting coefficient of 0.078 in Eq. (36), all have broad and unambiguous support for heat transfer, for which, however, reliable data extend only up to about $\mathrm{Pr} = 100$. Some of the experimental data for mass transfer, which extend up to $\mathrm{Sc} = 10^4$ are supportive of Eq. (36) but some, and in particular those from electrochemical measurements, are not. The recent Lagrangian direct numerical simulations by Papavassiliou and Hanratty [20] indicate reasonable agreement with the postulate of a limiting value of $\mathrm{Pr_t}$ as $y^+ \to 0$ for moderately large values of Pr but suggest that as $\mathrm{Pr} \to \infty$ the following dependence develops:

$$\mathrm{Pr_t} = 1.51(y^+)^{-0.38} \tag{38}$$

Eq. (38) is in serious conflict with the postulates made in deriving Eqs. (35) and (36). It leads to the following alternative to Eq. (36) in terms of heat transfer:

$$\mathrm{Nu} = 0.0889\mathrm{Re}_D\left(\frac{f}{2}\right)^{1/2}\mathrm{Pr}^{0.295} \tag{39}$$

Eq. (39) agrees with those very sets of experimental data for mass transfer for which Eq. (36) was noted to fail. Until this discrepancy is resolved by further work, Eqs. (35) and (36) are recommended for $\mathrm{Pr} < 100$ but not for larger values.

3.4. Error due to the use of Eq. (37) for general values of Pr

There is currently no exact criterion for evaluation of the magnitude of the error in Nu_D as a result of using Eq. (37) for Pr_t in Eq. (22) for general values of Pr. However, the consistency of the computed values of Nu_D with those for $Pr = 0$, $Pr = Pr_t = Pr_T$ and $Pr \to \infty$ and with prior theoretical values is examined in the next section.

4. Comparisons with prior results

The postulate that $j/j_w = 1$ or that $j/j_w = \tau/\tau_w = R$, and the uncertainty in the postulated velocity distribution, the eddy viscosity and the turbulent Prandtl number are the principal sources of error in prior theoretical predictions of the Nusselt number. Only those predictions in which the heat flux density ratio was taken into account exactly were considered for quantitative comparison with the results obtained herein. The two chosen sets of results for comparison are both based on the eddy viscosity and incorporate the velocity distribution separately and explicitly. The eddy viscosity and the velocity distribution are not inherently a source of uncertainty but the correlating equations that were utilized for these quantities are considered to be a significantly greater source of error in Nu_D than Eq. (6), which was utilized herein.

The solutions of Kays and Leung [21] for circular annuli, which include a uniformly heated round tube as a limiting case, were carried out by numerical integration of a partial differential energy balance using separate and thereby inconsistent empirical expressions for the eddy viscosity and the velocity distribution as well as one for the turbulent Prandtl number. The influence of their expressions for the turbulent Prandtl number and for the eddy viscosity would be expected to phase out as $Pr \to 0$, and the influence of their expression for the turbulent Prandtl number would be expected to be minimal on the interpolated value of their results for $Pr = 0.867$. The small differences in Nu_D in Table 2 are therefore presumed to be due to inaccuracy in their expressions for the velocity distribution and those in Table 3 to some extent to the inaccuracy of their expressions for the eddy viscosity and the turbulent Prandtl number as well.

Notter and Sleicher [18] determined values of Nu_D for both uniform wall temperature and uniform heating from a Graetz-type expansion in series. They utilized empirical correlating equations for the velocity, eddy viscosity and turbulent Prandtl number that are almost certainly more accurate than those used by Kays and Leung. However, the remarks concerning the discrepancies in the results of Kays and Leung for $Pr = 0$ and the interpolated values for $Pr = 0.867$ are applicable qualitatively to those of Notter and Sleicher that are included in Tables 2 and 3.

Balzhiser and Churchill [22] determined and presented graphically values of j/j_w for uniform heating at $Re_D = 4 \times 10^3$ and 2.35×10^6 in smooth pipe and for $Re_D = 2.5 \times 10^6$ in rough pipe with $a/e = 15$, using Eq. (15) and the classical correlating equations of that time period for the velocity distributions. Although a precise comparison is not possible, their plotted values appear to be in reasonable accord with those in Table 1.

All of these prior results for Nu_D and γ are presumed to be less accurate than those determined herein, primarily because of the superiority of Eq. (6) vis-à-vis the various correlating equations for the velocity and the eddy viscosity. Although the new computed values in Table 4 are of unknown accuracy they are presumed to be more accurate than prior values since the values used for Pr_t were the only significant source of error.

Finally, some generalizations may be stated concerning the prior theoretical results that were excluded from specific examination herein. The most common and serious error is that due to the postulate that $j/j_w = \tau/\tau_w = R$. The magnitude of this error for the particular conditions for which numerical predictions were made herein is represented quantitatively by ε_1 and ε_2 in Tables 2 and 3, respectively. The result of the idealization of $j/j_w = R$ is seen to be a significant overprediction of Nu_D. The other common and significant error is the postulate that $Pr_t = 1$ for all conditions. This postulate is implicit in most of the classical algebraic analogies (see, for example, Churchill [14]). The result is an underprediction of Nu_D by a factor that varies from approximately $(0.85)^{1/3} = 0.95$ for very large values of Pr to approximately 0.87 for Pr of the order of magnitude of unity. This error may compensate to some extent for that due to the postulate that $j/j_w = R$.

5. Summary and conclusions

In a recent article, Churchill [1] demonstrated that expression of the time-averaged momentum balance for fully developed flow in a round tube in terms of $(\overline{u'v'})^{++} \equiv -\rho\overline{u'v'}/\tau$, the local fraction of the shear stress due to turbulence, allows the formulation of a simple but exact integral expression for $u^+\{y^+, a^+\}$ and, by virtue of analytical integration by parts, another simple but exact integral expression for $u_m^+ = (2/f)^{1/2}$. Furthermore, he demonstrated that an analogous expression of the time-averaged energy balance for fully developed convection allows the formulation of corresponding exact integrals for T^+ and $T_m^+ = 2a^+/Nu_D$ but introduces two additional parameters, namely γ and Pr_t. Here $\gamma \equiv j\tau_w/j_w\tau - 1 = j/j_wR - 1$ is a measurement of the deviation of the local total heat flux density ratio from the local, total shear stress ratio, which varies linearly across the tube, and Pr_t is a symbol for $Pr((1/(\overline{T'v'})^{++}) - 1)/((1/(\overline{u'v'})^{++}) - 1)$, where $(\overline{T'v'})^{++} \equiv \rho c\overline{T'v'}/j$, is the local fraction of the heat flux density due to turbulence.

Churchill and Chan [6] devised an empirical but quite accurate correlating equation for $(\overline{u'v'})^{+} = (1 - (y^+/a^+))$

$(\overline{u'v'})^{++}$ and utilized it to determine essentially exact values for $u_m^+ = (2/f)^{1/2}$ for a moderate range of values of $a^+ = \mathrm{Re}_D/2u_m^+$. The great improvement in accuracy of u_m^+ with respect to $(\overline{u'v'})^{++}$ is a consequence of the process of smoothing that is inherent in both the original two successive integrations and the final consolidation to one. Their correlating equation for $(\overline{u'v'})^{+}$ and the consequent one for u_m^+ have been updated for use in the computations herein by virtue of the recent experimental time-mean velocity distributions of Zagarola [12].

A correlating equation is not required for γ since for uniform heating it may be expressed in terms of exact integrals of $(\overline{u'v'})^{++}$.

The currently available correlating equations and approximate theoretical expressions for $\mathrm{Pr_t}$ are quite uncertain, particularly for $\mathrm{Pr} < 0.7$. Fortunately, the dependence on Pr drops completely out of the formulations for Nu_D for two particular values of Pr, namely $\mathrm{Pr} = 0$ and $\mathrm{Pr} = \mathrm{Pr_t} \cong 0.867$. Churchill [1] was apparently the first to note the unique merit of the latter condition although the postulate of $\mathrm{Pr} = \mathrm{Pr_t}$ has often been made erroneously for all conditions, or particularly for $\mathrm{Pr} = 1$.

The numerical values of Nu_D presented herein for $\mathrm{Pr} = 0$ and $\mathrm{Pr} = \mathrm{Pr_t}$=0.867 are presumed to be essentially exact for the same reasons as mentioned for the previously computed values of $u_m^+ = (2/f)^{1/2}$. Despite their limitation to these two values of Pr and despite the relatively small improvement upon the best previously computed values, these results are invaluable as criteria for evaluation of experimental data, correlating equations and approximate theoretical values in the very range of conditions for which the uncertainty of the latter is greatest. For example, the values of Nu_D for $\mathrm{Pr} = 0$ constitute absolute lower bounds, while those for $\mathrm{Pr} = \mathrm{Pr_t}$ reveal that the classical algebraic analogies, with only a few exceptions, are in error by as much as 20% for $\mathrm{Pr} = 1$.

The formulations utilized herein could have been expressed in terms of the eddy viscosity rather than in terms of $(\overline{u'v'})^{++}$ with no loss of generality. However, the simplifications, and in particular the possibility of analytical integration by parts for u_m^+ and T_m' have never been recognized in such formulations and they have therefore all been explicit in terms of the velocity as well as the eddy viscosity. These new integral formulations require considerably less computation for a given degree of accuracy than numerical integration of the partial differential representation or the computation of eigenvalues and eigencoefficients. Results similar to those presented herein are readily attainable for all one-dimensional flows including those in parallel-plate channels and circular annuli.

Finally, it is remarkable that essentially exact numerical results may be computed for turbulent convection for $\mathrm{Pr} \to 0$, $\mathrm{Pr} \cong 0.87$ and $\mathrm{Pr} \to \infty$. The extension of such computations for other values of the Prandtl number depends on the development of better correlating equations or predictive methods for $\mathrm{Pr_t}$. It may be noted that this quantity appears to be a function only of $(\overline{u'v'})^{++}$ and Pr and thereby independent of geometry and the thermal boundary condition.

6. Nomenclature

a	radius of tube, m
a^+	$a(\tau_w\,\rho)^{1/2}/\mu$
c	specific heat capacity, J/kg K
D	diameter of tube, m
f	Fanning friction factor$=2\tau_w/\rho u_m^2$
h	heat transfer coefficient $j_w/(T_w - T_m)$, W/m^2 K
j	local time-averaged heat flux density in the negative radial direction, W/m^2
j_w	heat flux density from wall, W/m^2
k	thermal conductivity, W/m K
k_t	eddy thermal conductivity, W/m K
k_T	total thermal conductivity, $k + k_t$, W/m K
Nu_D	Nusselt number $= hD/k$
Pr	Prandtl number $= c\mu/k$
$\mathrm{Pr_t}$	turbulent Prandtl number $= c\mu_t/k_t$
$\mathrm{Pr_T}$	total Prandtl number $= c(\mu + \mu_t)/(k + k_t) = c\mu_T/k_T$
r	radial coordinate, m
R	dimensionless radius $= r/a$
Re_D	Reynolds number $= Du_m\rho/\mu$
T	time-averaged temperature, K
T'	fluctuation in temperature, K
T^+	dimensionless temperature $=k(\tau_w\rho)^{1/2}(T_w - T)/\mu j_w$
$(\overline{T'v'})^{++}$	dimensionless turbulent heat flux density$=\rho c\overline{T'v'}/j$
u	time-averaged longitudinal component of velocity, m/s
u^+	dimensionless longitudinal component of velocity $= u(\rho/\tau_w)^{1/2}$
u'	fluctuation in longitudinal component of velocity, m/s
$(\overline{u'v'})^{++}$	dimensionless turbulent shear stress$=-\rho\overline{u'v'}/\tau$
v'	fluctuation in (negative) radial component of velocity, m/s
y	distance from wall, m
y^+	dimensionless distance from wall $= y(\tau_w\rho)^{1/2}/\mu$

Greek symbols

α	arbitrary dimensionless coefficient of Eq. (34)
γ	$\frac{j}{j_w}\left(\frac{\tau_w}{\tau}\right) - 1$
ε_1	correction factor defined by Eq. (24)
ε_2	correction factor defined by Eq. (31)
μ	dynamic viscosity, kg/m s
μ_t	eddy viscosity, kg/m s
μ_T	total viscosity $= \mu + \mu_t$, kg/m s
ρ	specific density, kg/m^3
τ	total shear stress, Pa

Subscripts

c	at centerline
m	mixed-mean
w	at wall

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